A NOTE ON A CYCLICAL MAJORITY PROBLEM

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The purpose of this paper is to consider the "dilemma" imposed by the cyclical majority problem in which an imputation set like (½, ½, 0), (½, 0, ½), (0, ½, ½) precludes obtaining a single peaked preference set.[1] It will be suggested that the cyclically unstable outcome associated with such problems as how to divide one dollar among three persons operating with a majority rule is more of a "mathematical dilemma" than one ineluctably produced by individual utility maximizing behavior.

Assume that individuals A, B, and C are granted one dollar on the condition that it be divided in some way among them by majority rule. The impasse represented by the non-single peaked preference set seems to be the result of the denial of rational utility-maximizing behavior rather than the outcome of such behavior. The deadlock illustrated by the three-person F-set shown above results when the observer plots *ex post* the preference curves of individuals who are implicity assumed to behave in a special way. Once one introduces communication into the model, outcomes represented by the "solution"[2] imputations seem unlikely. Even if the vote on the division of the dollar is to be a one-time proposition, realistically there must be some period prior to the voting in which some three-persons bargaining can occur. There is a useful insight to be gained in viewing the preference curves in an *ex ante* sense and from the view of each individual.

In order to eliminate differential bargaining abilities, assume the three persons to be identical. Consider how each individual might draw his own preference curve given his knowledge of the situation. Suppose that each person will "vote" by writing on a slip of paper a ranking which shows how he prefers the money to be divided. The F-set requirements specify that a knowing A would rank this way: (½, ½, 0) or (½, 0, ½) first and (0, ½, ½) last. Of course, when B and C do likewise the preference set exhibits non-single peakedness--when the votes are counted they are all ties. But to suppose that each individual will actually vote this way when permitted time to communicate with every other individual may be absurd.

Communication permits the individuals to coordinate their choices, and this means that each can make his choice contingent on those of the other two. The essential purpose of such coordination in voting is to use the common interests of the three parties as leverage in their settlement of conflicting interests. It is true that A and B could connive and vote this way: A(½, ½, 0), B(½, ½, 0). This, of course, is one of three possible outcomes which comprise a set displaying non-single peakedness. Thinking in an *ex ante* sense, however, it may be assumed that each individual believes that any one of the three F-set outcomes is as likely as any other, so that each individual faces an expected pay-off of $ .33, i.e., he expects to "win" $ .50 two-thirds of the time. A reasonably weak assumption is that each individual is, ceteris paribus, a risk-averter; he will prefer $ .33 with certainity to $ .50 with a probability of success of two-thirds. With coordination, rational behavior should produce a (1/3, 1/3, 1/3) imputation which would secure majority agreement.[3] A reasonalbe approximation to the preference ordering of the three individuals is shown in the *ex ante* directed scales below.

A	B	C
(½,½,0) (½,0,½)	(½,½,0) (0,½,½)	(½,0,½) (0,½,½)
(1/3,1/3,1/3)	(1/3,1/3,1/3)	(1/3,1/3,1/3)
E [(½,½,0) (½,0,½)]=1/3	E [(½,½,0) (0,½,½)] = 1/3	E [(½,0,½) (0,½,½,)] =1/3
(0,½,½)	(½,0,½)	(½,½,0)

Individual A prefers (½, ½, 0) or (½, 0, ½) above all other outcomes, but he cannot be certain of achieving such an outcome. In fact, his probability of achieving one of the F-set imputations is only two-thirds, and this reduces the expected value of his voting that way to only $ .33. Under such circumstances, the rational risk-averter will prefer the symmetric sharing scheme of (1/3, 1/3, 1/3) yielding $ .33 with certainity to the expected value of obtaining one of the more preferred F-set outcomes. His fourth choice, (0, ½, ½) is the one where he receives no money at all. Individual A will therefore be motivated to vote for the (1/3, 1/3, 1/3) imputation. Since each individual will do likewise, that outcome is the one which will be adopted.

The (1/3, 1/3, 1/3) imputation has been criticized as being unstable because any majority can upset it. The arithmetical possiblity of beating the "equitable" imputation is irrelevant, however, unless the individuals *want* to beat it. And prior to the vote, when the individuals are mentally drawing their own preference curves in the light of their mutual circumstances, opting for a "solution" set imputation with its inherent uncertainty seems irrational. Individual A would choose (½, ½, 0) or (½, 0, ½) over (1/3, 1/3, 1/3) *if* he did not have to worry about (0, ½, ½). The rational risk-averter will bargain collectively to secure three-person symmetric sharing when the alternative is two-person sharing with a probability of success of two-thirds. This mathematical equivalence holds for large numbers as well although there may be communication and bargaining problems which could alter each individual's expectation of successful coalition and thus the expected value of a majority-sharing outcome.

There are circumstances, however, when one cannot even say that the (1/3, 1/3, 1/3) imputation is genuinely certain. It is quite possible that one of the individuals in the group will feel that he has a certainty for the (1/3, 1/3, 1/3) imputation because he is being deceived by the other two parties. Nevertheless, there are clearly cases in which an individual would opt for whatever probability he estimates exists for the (1/3, 1/3, 1/3) imputation together with some other probability under which he gets zero as against the two-thirds probability of getting one-half and the one-third probability of getting zero. Obviously, if the individual is certain he can be part of the winning coalition, he will choose one of the majority-sharing outcomes--presumably one of the F-set imputations. The choice confronting an individual like A, however, is not between alternative (1/3, 1/3, 1/3) and either (½, ½, 0) or (½, 0, ½). The other alternative, (0, ½, ½), is not an irrelevant alternative; it affects the choice considered in the previous sentence and leads to a preference ranking like that shown in the directed scales above. What seems wrong about the usage of the typical preference curve set diagram is that it abstracts from considering all the prospective alternatives.

FOOTNOTES

1. Duncan Black, THE THEORY OF COMMITTEES AND ELECTIONS. Cambridge: Cambridge University Press, 1958.

2. Only the set of imputations given in the first paragraph will satisfy the Von Neuman-Morgenstern requirements for "solution" to n-person games. The first requirement is that no single imputation in the F-set either dominates or is dominated by any other imputation in that set. The second requirement is that any imputation not not in F is dominated by at least one imputation in F. See, for example, the three-person, constant-sum game analyzed by James M. Buchanan and Gordon Tullock in Chapter II of THE CALCULUS OF CONSENT.

3. Actually only two of the individuals' need prefer $ .33 with certainty to $ .50 with P (.67).